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Simple Harmonic Oscillator:

The potential for a harmonic oscillator is:

$V = \frac{1}{2} m\omega^2 x^2$ ω: natural frequency

The importance of harmonic oscillator is that small fluctuations of a system near its point of equilibrium follows $V = \frac{1}{2} m\omega^2 \eta^2$:

$V'(\eta_0) = 0 \Rightarrow V(\eta_0 + \delta\eta) \approx V(\eta_0) + \frac{1}{2} V''(\eta_0) \delta\eta^2$

If $V''(\eta_0) > 0$, equilibrium point is stable and for small $\delta\eta$ the system undergoes harmonic motion.

The equation of motion for classical oscillator is:

$\ddot{\eta} + \omega^2 \eta = 0 \Rightarrow \eta = A \sin\omega t + B \cos\omega t$

It is easy to see that the time-averaged kinetic and potential energy are equal, as expected for q^2 potential from Virial's theorem.

$$\langle T\rangle = \frac{1}{2} m\omega^2 \langle (A\cos\omega t - B\sin\omega t)^2\rangle = \frac{1}{4} m\omega^2 (A^2+B^2)$$

$$\langle V\rangle = \frac{1}{2} m\omega^2 \langle (A\sin\omega t + B\cos\omega t)^2\rangle = \frac{1}{4} m\omega^2 (A^2+B^2)$$

$$E = T + V = \frac{1}{2} m\omega^2 (A^2+B^2)$$

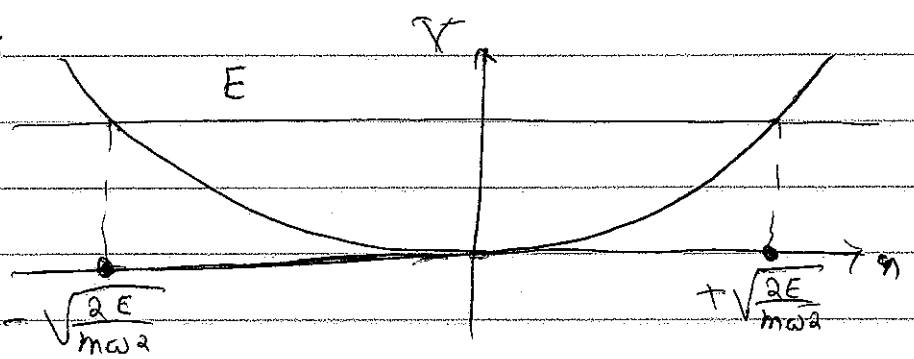
Now lets turn to quantum oscillator. The eigenvalue problem in this case is:

$$H|E\rangle = E|E\rangle \qquad H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 X^2$$

Where in the eigen basis of x:

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2$$

First of all, we note that the energy eigenstates are all bound states. No matter how large E is, at sufficiently large distance $|x| > \sqrt{\frac{2E}{m\omega^2}}$ we have $E < V$:



Therefore the spectrum of harmonic oscillator is discrete.

Moreover, all the energy eigenvalues are positive. For an arbitrary state vector $|\psi\rangle$, we have:

$$\langle H\rangle = \langle\psi|\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2|\psi\rangle = \langle\psi|\frac{p^2}{2m}|\psi\rangle + \langle\psi|\frac{1}{2}m\omega^2x^2|\psi\rangle = \frac{1}{2m}\langle p\psi|p\psi\rangle + \frac{1}{2}m\omega^2\langle x\psi|x\psi\rangle$$

Each of these terms are positive definite in general. However, it is not possible that $x|\psi\rangle = p|\psi\rangle = 0$ at the same time. This comes from the commutation relation between x and p:

$$[x,p] = i\hbar \Rightarrow xp|\psi\rangle - px|\psi\rangle = i\hbar|\psi\rangle$$

If $x|\psi\rangle = p|\psi\rangle = 0$, then the left-hand side is zero, while the right-hand side is not. We therefore conclude that $\langle H\rangle > 0$, implying that the smallest energy eigenvalue (i.e. ground

state energy) is positive.

Since the Hamiltonian is quadratic in both p and x, it suggests that the eigenvalue problem may have a solution that has the same form in x and p eigenbases. Namely, a Gaussian wavepacket. Indeed, we can check that $\psi(x) = \left(\frac{\pi}{\alpha}\right)^{-\frac{1}{4}} \exp\left(-\frac{\alpha}{2}x^2\right)$ is a solution, where:

$$\alpha = \frac{m\omega}{\hbar}, \quad E = \frac{1}{2}\hbar\omega$$

As it turns out, this is the ground state of harmonic oscillator.

In the x eigenbasis, the energy eigenstates are given by (detaile derivation in the book):

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar^2 2^{2n}(n!)^2}\right)^{\frac{1}{4}} \exp\left(\frac{-m\omega x^2}{2\hbar}\right) H_n\left[\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}} x\right]$$

The corresponding energy eigenvalues are $E_n = (n+\frac{1}{2})\hbar\omega$,

and $n=0,1,\ldots$. H_n are Hermite polynomials, that satisfy the orthonormality condition:

$$\int_{-\infty}^{+\infty} H_n(y)\, H_m(y)\, e^{-y^2}\, dy = \delta_{nm}\left(\sqrt{\pi}\, 2^n\, n!\right)$$

$H_n(y)$ are even functions for $n=0,2,\ldots$ and odd functions for $n=1,3,\ldots$. Therefore energy eigenstates are Hermite polynomials with Gaussian envelope.

Energy Basis:

One can define two operators $a, a^\dagger$ according to:

$$a = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}} X + i\left(\frac{1}{2\omega m\hbar}\right)^{\frac{1}{2}} P$$

$$a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}} X - i\left(\frac{1}{2m\omega\hbar}\right)^{\frac{1}{2}} P$$

They satisfy the commutation relation:

$$[a, a^\dagger] = 1$$

And we have:

$H = (a^\dagger a + \frac{1}{2})\hbar\omega$

Consider an energy eigenstate $|\epsilon\rangle$ such that $H|\epsilon\rangle = \hbar\omega(\epsilon + \frac{1}{2})|\epsilon\rangle$. Then we find:

$H(a|\epsilon\rangle) = \hbar\omega(\epsilon - \frac{1}{2})a|\epsilon\rangle$, $H(a^\dagger|\epsilon\rangle) = \hbar\omega(\epsilon + \frac{3}{2})a^\dagger|\epsilon\rangle$

Thus:

$a|\epsilon\rangle = C_\epsilon|\epsilon-1\rangle$, $a^\dagger|\epsilon\rangle = C_\epsilon|\epsilon+1\rangle$

C_ϵ, $C_{\epsilon+1}$ are constants that we will find later.

Hence if $|\epsilon\rangle$ is an eigenstate with eigenvalue $\hbar\omega(\epsilon + \frac{1}{2})$, the $a|\epsilon\rangle$ $(a^\dagger|\epsilon\rangle)$ will be an eigenstate with eigenvalue $\hbar\omega(\epsilon-1+\frac{1}{2})$ $(\hbar\omega(\epsilon+1+\frac{1}{2}))$

For this reason we call a and $a^\dagger$ lowering and raising operators respectively.